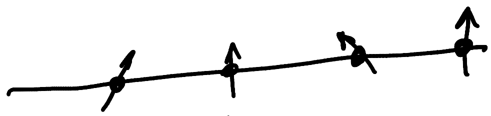
Jordan-Wigner transformation

Consider a 1D spin model



for example,

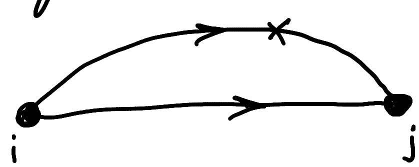
$$\hat{H} = \sum_i \left(J_x \sigma_i^x \sigma_{i+1}^x + J_y \sigma_i^y \sigma_{i+1}^y + J_z \sigma_i^z \sigma_{i+1}^z - B \sigma_i^z \right) \quad (*)$$

(anisotropic Heisenberg model)

$\forall$ spin $\exists$ a basis of two states: $\uparrow$ and $\downarrow$

Can we map that onto a system of fermions on a system of sites? Fermions on a site have also got two states: $n_i = 0$ and $n_i = 1$. Looks like there should be a mapping.

There is a problem, however. Fermionic operators anticommute on different sites, $\{\hat{a}_i, \hat{a}_j\} = 0$, while spin operators commute, $[\hat{S}_i, \hat{S}_j] = 0$. This means fermions pick up extra phases. These phases do not matter, however, and may be gauged out in 1D, because there is only one way to go from a site to site.



Jordan-Wigner transformation

$\hat{\sigma}^{-} = \hat{\sigma}_i^x - i\hat{\sigma}_i^y \quad \hat{a} \quad \prod \hat{\sigma}^z$

Jordan - Wigner transformation

$$\hat{\sigma}_i^z = \underbrace{2\hat{a}_i^+\hat{a}_i - 1}_{\text{Values } \pm 1 \text{ as necessary}}, \qquad \hat{\sigma}_i^- \equiv \frac{\hat{\sigma}_i^x - i\hat{\sigma}_i^y}{2} = \hat{a}_i \underbrace{\prod_{j<i} \hat{\sigma}_j^z}_{\text{„Corrects" the phase}}$$

(Recall $\langle \ldots, 0, \ldots | \hat{a}_i | \ldots, 1, \ldots \rangle = (-1)^{\sum_{l=1}^{i-1} n_l}$)

$$\hat{\sigma}_i^+ = \hat{a}_i^+ \prod_{j<i} \hat{\sigma}_j^z$$

The reverse transformation:

$$\hat{a}_i = \hat{\sigma}_i^- \prod_{j<i} \hat{\sigma}_i^z$$

$$\hat{a}_i^+ = \hat{\sigma}_i^+ \prod_{j<i} \hat{\sigma}_i^z$$

Express the initial Hamiltonian (*) of the Heisenberg model in the fermionic operators

$$\sigma_i^x \sigma_{i+1}^x = (\sigma_i^+ + \sigma_i^-)(\sigma_{i+1}^+ + \sigma_{i+1}^-) =$$

$$= (a_{i+1}^+ + a_{i+1}) \underbrace{\sigma_i^z}_{=2a_i^+ a_i - 1} (a_i^+ + a_i) =$$

$$= a_{i+1}^+ a_i^+ + a_{i+1} a_i^+ - a_{i+1}^+ a_i - a_{i+1} a_i$$

The other terms are done similarly

$$\hat{H} = \sum_i \Big[-(J_x + J_y)\, \hat{a}_i^+ \hat{a}_{i+1} + (J_x - J_y)\, \hat{a}_i \hat{a}_{i+1} + h.c.$$

$$+ J_z (2\underbrace{n_i}_{=a_i^+ a_i} - 1)(2n_{i+1} - 1) - B(2n_i - 1) \Big]$$

It is often convenient to solve 1D problems in the fermionic representation

Example:

$J_x = J_y = -\frac{1}{2} J < 0 \ , \ J_z = 0$

(a ferromagnetic chain)

$$\hat{H} = -\frac{1}{2} J \sum_i \left(\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y \right) - \sum_i B \sigma_i^z$$

Mapping onto fermions

$$\hat{H} = \sum_i \left[J \left(\hat{a}_i^+ \hat{a}_{i+1} + \hat{a}_{i+1} \hat{a}_i \right) - B \left(2\hat{n}_i - 1 \right) \right]$$

This is the Hamiltonian of fermions hopping on a lattice and having the energy $-2B$ on each site (the difference between $n_i = 0$ and $n_i = 1$)

$$\hat{a}_k = \frac{1}{\sqrt{N}} \sum_n \hat{a}_n e^{ikn}$$

In the case of an infinite lattice, $\hat{a}_k = \sum_n \hat{a}_n e^{ikn}$

$$\hat{H} = \int_{-\pi}^{\pi} 2 \left(J \cos k - B \right) \hat{a}_k^+ \hat{a}_k \frac{dk}{2\pi}$$

$$\varepsilon_k = 2 \left(J \cos k - B \right)$$ – the dispersion of the excitations

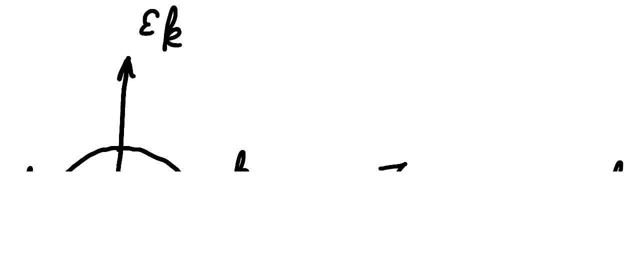
The average magnetisation per site

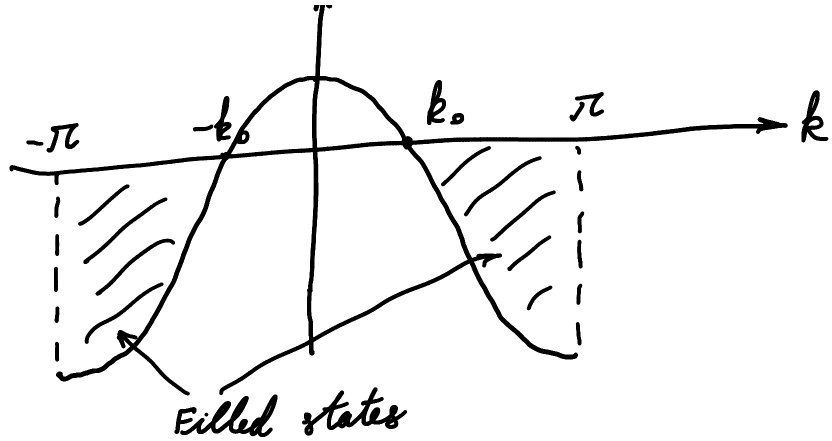
$$m = \langle \sigma_i^z \rangle = \langle 2 \hat{n}_i - 1 \rangle = \langle 2 a_i^+ a_i - 1 \rangle =$$

$$= \int_{-\pi}^{\pi} \left(2 \langle a_i^+ a_i \rangle - 1 \right) \frac{dk}{2\pi}$$

Assume $B > 0$

If $J > B$, then the system is partially magnetised at $T = 0$





$$k_0 = \frac{\pi}{2} - \arcsin\frac{B}{J}$$

The magnetisation

$$\mu = \frac{1}{2\pi}\left[-1\cdot 2k_0 + 1\cdot(2\pi - 2k_0)\right] = \frac{2}{\pi}\arcsin\frac{B}{J}$$

If $B > J$, the entire system is magnetised and $\mu = 1$

Magnetic susceptibility $\chi = \frac{\partial \mu}{\partial B} = \begin{cases} \frac{2}{\pi}\frac{1}{\sqrt{J^2 - B^2}} & , B < J \\ 0 & , B > J \end{cases}$